Compact quantum electrodynamics in 2+1 dimensions and spinon deconfinement: A renormalization group analysis

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We discuss compact 2+1 dimensional Maxwell electrodynamics coupled to fermionic matter with N replica. For large enough N, the latter corresponds to an effective theory for the nearest neighbor SU(N) Heisenberg antiferromagnet, in which the fermions represent solitonic excitations known as spinons. Here, we show that the spinons are deconfined for $N > N_c = 36$, thus leading to an insulating state known as spin liquid. A previous analysis considerably underestimated the value of N_c. We show further that for $20 < N \leq 36$, there can be either a confined or a deconfined phase, depending on the instanton density. For $N \leq 20$, only the confined phase exists. For the physically relevant value $N = 2$, we argue that no paramagnetic phase can emerge since chiral symmetry breaking would disrupt it. In such a case, a spin liquid or any other nontrivial paramagnetic state (for instance, a valence-bond solid) is only possible if doping or frustrating interactions are included.

I. INTRODUCTION

Quantum electrodynamics in $D+1$ space-time dimensions (QED$_{D,1}$) with $D = 1, 2$ are useful field-theoretic models in high-energy physics. Phenomena such as chiral symmetry breaking and confinement are easier to understand in QED$_{1,1}$ and QED$_{2,1}$ than in QCD. The simplest, exactly solvable model of this type is spinor quantum electrodynamics in the 1+1 space-time dimensions (QED$_{1,1}$), the so-called Schwinger model, which exhibits both chiral symmetry breaking and confinement.[1] Another example, relevant to the present paper, is the 2+1-dimensional spinor quantum electrodynamics, QED$_{2,1}$, in the form introduced by Pisarski some time ago.[2] This model is known to exhibit spontaneous chiral symmetry breaking.[3,4] An interesting aspect of this model is its applicability to condensed matter physics where it appears in different contexts, especially in the study of high-T_c superconductors and Mott insulators.[5–9] In the study of Mott insulators, it arises as an effective theory of the so-called *spin liquids*, which are Mott insulators without any broken symmetry. In this case, the Dirac fermions represent the so-called *spinons*, solitonlike excitations carrying spin degrees of freedom but no charge. A good name for this type of QED is *quantum spinodynamics* (QSD) since it is actually a quantum field theory of spinons. The theory can be derived for Mott insulators, and it is found that the Abelian gauge field coupling to the spinons is compact. This follows immediately by accounting for the fluctuations around mean-field theories of resonating valence bonds[10] states which have a local U(1) gauge freedom in which the phase angle is defined only modulo 2π. These mean-field theories are derived from the strong-coupling limit of the Hubbard model, the so-called Heisenberg-Hubbard model,

$$H = J \sum_{\langle i,j \rangle} \mathbf{S}_i \cdot \mathbf{S}_j, \tag{1}$$

where $\mathbf{S}_i$ are spin-$1/2$ operators formed from Fermi fields $f_{i\beta}$ subjected to the local constraint $f_{i\sigma}^\dagger f_{i\sigma} = 1$ as $\mathbf{S}_i = (1/2) f_{i\alpha}^\dagger \boldsymbol{\sigma}_{\alpha\beta} f_{i\beta}$, where $\boldsymbol{\sigma}$ are the Pauli matrices $\boldsymbol{\sigma} \equiv (\sigma_1, \sigma_2, \sigma_3)$ (throughout this paper summation over re-

peated Greek indices is implied). The sum in Eq. (1) runs over nearest neighbor pairs of sites in a square lattice, and J is related to the original parameters of hopping and interaction energies t and U of the Hubbard model by $J = 4t^2/U$. Different composite-field theories are known to represent the same quantum states of the model. This follows from the fact that the *local* gauge symmetry of the Heisenberg-Hubbard model is actually SU(2).[11] Thus, the composite link fields

$$\Delta_{ij}^* = \langle f_{i\uparrow}^\dagger f_{j\downarrow}^\dagger - f_{i\downarrow}^\dagger f_{j\uparrow}^\dagger \rangle, \tag{2}$$

and

$$\chi_{ij} = \langle f_{i\alpha}^\dagger f_{j\alpha} \rangle, \tag{3}$$

where (i, j) are nearest neighbors, which are obtained from different Hubbard-Stratonovich decouplings of the Heisenberg-Hubbard model, describe the *same* physics when associated with the most stable ground state of the corresponding mean-field theory. They are connected through a SU(2) gauge transformation.[11]

The phase fluctuations of either composite field features a lattice gauge field A_{ij}. These are obviously of the compact U(1) type. Both fields together transform into each other by local SU(2) transformations, and theories utilizing this symmetry have been studied in the past,[12,13] with some advantages over compact U(1) theories, especially if one is interested in studying the phase structure of high-T_c superconductors.[13–15] A local SU(2) gauge theory also emerges in the study of frustrated Heisenberg antiferromagnets.[16,17] The existence of the larger SU(2) symmetry ensures the compactness of the Abelian theories even in the continuum limit since the U(1) group is a subgroup of SU(2).[18]

A controlled study of U(1) spin liquids was initiated some time ago by Affleck and Marston[5] using the composite field [Eq. (3)]. To have a small expansion parameter, they generalized the *global* symmetry SU(2) to SU(N) and solved the model in the large N limit. This was done in the so-called self-conjugate representation of SU(N),[19,20] where the spin operators are given by $S_{i,\alpha\beta} = f_{i\alpha}^\dagger f_{i\beta} - \delta_{\alpha\beta}/2$ and fulfill the local constraint $f_{i\alpha}^\dagger f_{i\alpha} = N/2$. These operators have zero trace:

$\mathrm{Tr}(S_{i,\alpha\beta}) = 0$. If the partition function is represented as a functional integral over the action associated with the Hamiltonian [Eq. (1)], we may calculate a mean-field approximation from the saddle point approximation. The result which preserves all the lattice symmetries is the so-called π-flux phase,[5] whose spectrum of elementary excitations is given by

$$E_{\mathbf{k}} = 2|\chi_0|\sqrt{\cos^2 k_x + \cos^2 k_y}, \qquad (4)$$

where $|\chi_0|$ is the mean-field amplitude of χ_{ij}. The excitations around the Fermi points $\pm(\pi/2, \pi/2)$ can be at low energies represented as four-component Dirac fermions. The phases of χ_{ij} fluctuate strongly and form a link gauge field A_{ij}. It is not difficult to show that the effective low-energy Lagrangian in imaginary time has the form[21]

$$\mathcal{L} = \frac{1}{4e_0^2}F_{\mu\nu}^2 + \sum_{a=1}^{N}\bar{\psi}_a(\slashed{\partial} + i\slashed{A})\psi_a, \qquad (5)$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and we have used the usual Feynman slash notation $\slashed{a} \equiv \gamma_\mu a_\mu$, with γ_μ being 4×4 γ matrices. In the above continuum notation, the compactness of the gauge field is not apparent. As it stands, the above Lagrangian is just the above-mentioned massless $\mathrm{QED}_{2,1}$, which is a well studied model. We shall reserve the abbreviation QSD for the compact version of $\mathrm{QED}_{2,1}$, whose field theory will be discussed at length in Sec. III.

In QSD, the spinons play a role similar to quarks in QCD. Indeed, compactness of the gauge field leads to spinon confinement if N is not large enough.[6,22–24] Note, however, an important difference. In QCD, the gluon is introduced *ad hoc* to generate the coupling between the quarks. The "gluon" in QSD, on the other hand, has a clear origin: it is spontaneously generated by the phase fluctuations of the link field [Eq. (3)], which in turn is a composite field made of lattice fermions. This unique feature of QSD led Wen[25] to propose a similar mechanism in 3+1 dimensions to explain the origin of gauge bosons.

Whether spinons in QSD deconfine or not was for some time matter of controversy.[23,26–30] The controversy seems now to be solved,[22,31] at least at a qualitative level. The result is that spinons deconfine for large enough N,[22] and this guarantees the stability of the large-N spin liquid. The aim of this paper is to further improve our understanding of spinon deconfinement, in particular, by a more quantitative analysis based on the renormalization group (RG). We start in Sec. II with a discussion of the model in the absence of matter, which just corresponds to the compact version of Maxwell electrodynamics in 2+1 dimensions ($\overline{\mathrm{QED}}_{2,1}$).[18] The equivalence of this model with a three-dimensional Coulomb gas of instantons allows us to make a relatively simple field theoretical analysis, based on the equivalence of a Coulomb gas with the sine-Gordon theory. In Sec. II, we compute the one-loop effective potential of the model in terms of a scalar field $\varphi(x)$ whose correlation function $\langle\varphi(x)\varphi(x')\rangle$ gives a direct measure of the interaction between instantons in three-dimensional space-time. The calculation of the effective potential leads to an estimate of the instanton mass (which is different from the mass of the scalar field). We also derive the RG equations for the model. The resulting effective potential and the RG equations confirm in a physically appealing way the well-known result[18,32] that in absence of matter, the 2+1-dimensional compact gauge theory permanently confine test charges. This result contrasts with the one in 3+1 dimensions, where a deconfined phase has been proven to exist.[33–35]

Section III is where we start with the analysis of compact $\mathrm{QED}_{2,1}$ or QSD. Here, it is very difficult to write a field theory in terms of the original variables. It will be necessary, just like in Sec. II, to work with a field theory describing the dynamics of the instantons. We still have a gas of instantons, but it is no longer of the Coulomb type, since the interaction between the instantons is modified by the vacuum polarization. The RG equations are obtained in essentially the same way as in Sec. II, except that the effects of the vacuum polarization are taken into account. These change the phase structure of the theory in an essential way. By considering a theory with N replica of Dirac fermions, we show that spinons deconfine for $N > N_c = 36$. This is a considerable numerical change with respect with our previous calculation of N_c,[31] where a value smaller by a factor π^3 was found due to a wrong counting of π factors in the RG calculation. Apart from this mistake, the analysis in Ref. 31 is correct, in the sense that a critical value of N is predicted above which deconfinement occurs. Here, we elaborate further the nature of the confined phase. In particular, we show that confinement is the only possibility for $N \leq 20$, while deconfinement certainly occurs only for $N > 36$. In the region $20 < N \leq 36$, either phase is possible, depending on the instanton density. The universal properties of QSD are encoded in the coefficient of the $-1/R$ contribution to the interspinon potential. In the deconfined phase, this coefficient is given by the fixed point in the renormalization flow derived in the $1/N$ expansion. However, in the confined phase, the $1/N$ expansion does not hold and an expansion in the fugacity is made instead. Spinon deconfinement implies the stability of the spin liquid. The experimentally relevant case is the $N = 2$ one, which corresponds to the regime where the spin liquid is unstable since the spinons are confined in this case. The nature of the confined phase for $N = 2$ is discussed in Sec. IV, where the important role played by chiral symmetry breaking is emphasized.

II. COMPACT MAXWELL ELECTRODYNAMICS IN 2+1 DIMENSIONS

The 2+1-dimensional compact electrodynamics studied by Polyakov,[18] abbreviated here as $\overline{\mathrm{QED}}_{2,1}$, is actually Maxwell theory in 2+1 dimensions in which the gauge group U(1) is made compact. The model was originally motivated by the spontaneous symmetry breaking pattern of the Georgi-Glashow model in 2+1 dimensions. After spontaneous symmetry breaking of the SU(2) group, one is left with a residual U(1) group which is compact since it is a subgroup of SU(2). Another place where this theory arises naturally is in lattice gauge theory.[36]

It is well known that $\overline{\mathrm{QED}}_{2,1}$ is equivalent via duality to a three-dimensional Coulomb gas of instantons,[18] whose energy interaction is given by

$$E_I = \frac{2\pi^2}{e_0^2} \sum_{i,j} \frac{q_i q_j}{4\pi|x_i - x_j|}, \tag{6}$$

where e_0 is the bare gauge coupling and $q_i = \pm q$ with $q \in \mathbb{N}$ are the instanton charges.

In two dimensions, the Coulomb gas corresponds to a theory dual to the two-dimensional XY model[37,38] or, equivalently, a two-dimensional classical superfluid.[39] The Coulomb interaction between the charges is in this case proportional to $\ln|x_i - x_j|$. In the context of two-dimensional superfluids, the charges in the Coulomb gas are interpreted as vortices in two dimensions. The two-dimensional Coulomb gas is known to undergo a vortex-antivortex pair unbinding phase transition, the celebrated Kosterlitz-Thouless (KT) phase transition.[39] In the language of the Coulomb gas, we have a low-temperature dielectric phase separated from the high-temperature "metallic" plasma phase by a phase transition without breaking the U(1) symmetry of the original superfluid or XY system, in agreement with the Mermin-Wagner theorem.[40] In three dimensions, on the other hand, no phase transition occurs in the Coulomb gas and the system remains in the plasma phase. In this case, the Debye-Hückel (DH) mean-field theory is essentially correct and the interaction is screened in such a way that the excitations are always gapped. This can be conveniently expressed in field theory language by means of the sine-Gordon representation of the Coulomb gas. In the context of $\overline{\text{QED}}_{2,1}$, the corresponding sine-Gordon theory reads

$$\mathcal{L}_{\text{SG}} = \frac{1}{2}(\partial_\mu \varphi)^2 - 2z_0 \cos\left(\frac{2\pi}{e_0}\varphi\right). \tag{7}$$

The above Lagrangian corresponds to the field theory model dual to $\overline{\text{QED}}_{2,1}$. The parameter z_0 is the bare fugacity of the Coulomb gas. The DH theory amounts to a Gaussian approximation to the above Lagrangian. This leads to a mass $M_0 = 2\pi\sqrt{2z_0}/e_0$ for the scalar field φ. This behavior of the dual model implies a corresponding mass gap in the magnetic field correlation function.[18] Fluctuation corrections to the DH approximation essentially do not change this result. To see this, let us compute the one-loop effective potential. This can be easily obtained by standard methods.[41] At one-loop order, it is more easily obtained by writing $\varphi = \bar{\varphi} + \delta\varphi$, where $\bar{\varphi}$ is a constant background field, while $\delta\varphi$ represents a small fluctuation around it. By integrating out the Gaussian fluctuations, the one-loop effective potential is obtained,

$$V_{\text{eff}}(\bar{\varphi}) = -2z_0 \cos\left(\frac{2\pi}{e_0}\bar{\varphi}\right) - \frac{2\pi^2}{3}\left[\frac{z_0}{e_0^2}\cos\left(\frac{2\pi}{e_0}\bar{\varphi}\right)\right]^{3/2}, \tag{8}$$

where a counterterm proportional to $\cos(2\pi\varphi/e_0)$ was used to trivially subtract the contribution $(2z_0\Lambda/e_0^2)\cos(2\pi\varphi/e_0)$, with Λ being an ultraviolet cutoff. As with Eq. (7), the obtained effective potential implies a degenerate vacuum at $\bar{\varphi}_n = ne_0$, $n \in \mathbb{Z}$, whose energy density is given by $E_0 = -2z_0 - (2\pi^2/3)(z_0/e_0^2)^{3/2}$, which is always negative, just as the energy of the vacuum without the quantum corrections. From

this, we conclude that the instantons of this theory are always massive.

In order to better understand the meaning of this statement, let us compare the above effective potential with the one of a $1+1$-dimensional sine-Gordon theory. In this case, the corresponding Euclidean theory is the dual field theory of a two-dimensional superfluid,[37] which is known to undergo a KT phase transition.[39] The one-loop effective potential reads[42]

$$V_{\text{eff}}(\bar{\varphi}) = -2z\cos(2\pi\sqrt{K}\bar{\varphi}) - \pi Kz\cos(2\pi\sqrt{K}\bar{\varphi})\ln\left[\frac{4\pi^2 Kz}{\Lambda^2}\cos(2\pi\sqrt{K}\bar{\varphi})\right], \tag{9}$$

where K is the superfluid stiffness and Λ an ultraviolet cutoff. In order to remove the cutoff, we add a counterterm proportional to $\cos(2\pi\sqrt{K}\bar{\varphi})$ and impose the renormalization condition $V_{\text{eff}}''(0) = 8\pi^2 Kz \equiv M_\varphi^2$, which just fixes the renormalized mass of the scalar field in such a way as to have the same form as the one obtained from the DH theory. The result is

$$V_{\text{eff}}(\bar{\varphi}) = -z(2 - \pi K)\cos(2\pi\sqrt{K}\bar{\varphi}) - \pi Kz\cos(2\pi\sqrt{K}\bar{\varphi})\ln[\cos(2\pi\sqrt{K}\bar{\varphi})]. \tag{10}$$

The ground state energy density is now $E_0 = z(\pi K - 2)$. This changes sign at $K_c = 2/\pi$, precisely at the critical KT stiffness value.[37] In this case, we can use the results of Ref. 43 to obtain the soliton mass as

$$M_{\text{sol}} = \frac{2}{\pi}\sqrt{\frac{2z}{K}}(2 - \pi K). \tag{11}$$

For $K \gtrsim K_c$, which corresponds to low temperatures in the KT theory, the soliton mass vanishes while the energy becomes positive. For $K < K_c$, on the other hand, we can write

$$M_{\text{sol}} = -\frac{2}{\pi}\sqrt{\frac{2}{zK}}E_0 = -\frac{8E_0}{M_\varphi}. \tag{12}$$

Note that while the soliton mass vanishes at K_c, the scalar field mass does not. This is very important, since in the $2+1$-dimensional case the mass of the scalar field also does not vanish for any e_0^2, but there the same is true of the mass of instanton excitations. By analogy to Eq. (12), we can infer that the instanton mass in CMT_2 should be given by

$$M_{\text{inst}} \propto -\frac{E_0}{M_0} = \frac{e_0}{\pi\sqrt{2z_0}}\left[z_0 + \frac{\pi^2}{3}\left(\frac{z_0}{e_0^2}\right)^{3/2}\right], \tag{13}$$

which never vanishes in contrast to Eq. (12).

The above results are made more transparent when we consider the RG equations of $\overline{\text{QED}}_3$, which are equivalent to the RG equations of the three-dimensional Coulomb gas. To lowest order, this may be achieved by means of a scale-dependent DH approximation, in the same spirit of the analysis of the two-dimensional Coulomb gas made by Young.[44] The RG equations for the three-dimensional Coulomb gas were originally derived by Kosterlitz[45] using the so-called

poor-man scaling approach. As shall we see, for our purpose, it is better to use Young's approach,[44] which we generalize to the higher-dimensional case. The derivation is given in the Appendix for the case of a d-dimensional Coulomb gas. Again, it will be useful to compare the RG equations for the KT case with those for $\overline{\text{QED}}_{2,1}$. Setting $d=2$ in Eqs. (A17) and (A18) of the Appendix, we obtain the well-known RG equations for a two-dimensional superfluid where $\kappa=K$,

$$\frac{dK^{-1}}{dl} = y^2, \qquad (14)$$

$$\frac{dy}{dl} = (2 - \pi K)y. \qquad (15)$$

From these equations, we can see better why our approach of K_c from above drives the phase transition. In this case, the sign on the right-hand side of Eq. (15) is negative, such that the system will flow to a regime of zero fugacity, leading to a line of fixed points. At this line, the mass of the scalar field vanishes. If we define a dimensionless soliton mass through $m_{\text{sol}} \equiv M_{\text{sol}}/\Lambda$, we obtain

$$\frac{dm_{\text{sol}}^2}{dl} = \frac{2(2/\pi)^2}{K}[(2 - \pi K)^2 + 2y^2 K]. \qquad (16)$$

The fixed point of Eq. (16) corresponds precisely to the KT critical point. In this equation, it should be understood that for $y=0$ and $K \geq K_c$, its right-hand side vanishes since M_{sol} is zero for all $z \geq 0$ and $K \geq K_c$.

Now let us see what happens in $\overline{\text{QED}}_3$. In this case, we set $K_0 = a = 1/e_0^2$ in the equations of the Appendix, where a is a short-distance cutoff, and define the dimensionless gauge coupling $f \equiv K_0/K = 1/\kappa = e^2/e_0^2$, such that

$$f(l) = e_0^2 a e^l \varepsilon(ae^l) = e^l \varepsilon(ae^l), \qquad (17)$$

with $l \equiv \ln(r/a)$ being a logarithmic length scale. Thus, $f(l)e^{-l}$ is just given by the screening constant ("dielectric" constant) of the Coulomb gas of instantons. In this way, we obtain from Eqs. (A17) and (A18) the RG equations for the $\overline{\text{QED}}_3$,

$$\frac{df}{dl} = y^2 + f, \qquad (18)$$

$$\frac{dy}{dl} = \left(3 - \frac{\pi}{2f}\right)y, \qquad (19)$$

Note that the coefficient of the term $1/f$ in Eq. (19) is different from the one obtained by us in Ref. 31, where factors of π were overcounted. It is easy to see that, in contrast with Eqs. (14) and (15), the RG flow pattern does not exhibit any fixed point, giving a further confirmation that the three-dimensional sine-Gordon model does not undergo any phase transition. In other words, the excitations are always gapped.

III. COMPACT QUANTUM ELECTRODYNAMICS IN 2+1 DIMENSIONS (QUANTUM SPINODYNAMICS)

When fermionic matter is present, the energy of the instanton gas [Eq. (6)] is changed due to the vacuum polariza-

tion. The effective bare gauge coupling is modified to $e^2(p) = Z_A(p)e_0^2$, where $Z_A(p)$ is the wave function renormalization of the gauge field in the noncompact theory. We have

$$Z_A(p) = \frac{1}{1 + \Pi(p)}, \qquad (20)$$

where $\Pi(p)$ is the vacuum polarization. Thus, the energy of the instanton gas becomes

$$E_I = -\frac{1}{2}\sum_{i,j} U_0(x_i - x_j)q_i q_j, \qquad (21)$$

where

$$U_0(x) = -\frac{4\pi^2}{e_0^2}\int\frac{d^3p}{(2\pi)^3}\frac{e^{ip\cdot x}}{p^2}[1 + \Pi(p)] \qquad (22)$$

is the interaction between two instantons of opposite charge.

Let us emphasize the dual nature of the above potential by writing the *static* semiclassical interspinon potential between two opposite test spinon charges,

$$V(R) = -e_0^2\int\frac{d^2p}{(2\pi)^2}\frac{e^{i\mathbf{p}\cdot\mathbf{x}}}{\mathbf{p}^2[1 + \Pi(\mathbf{p})]}, \qquad (23)$$

where $R \equiv |\mathbf{x}|$. We see that there is a manifest duality between "electric" and instanton charges even after the vacuum polarization is included. Indeed, from Eq. (22), the effective squared instanton charge is given by $\tilde{e}^2(p) = 4\pi^2/e^2(p)$, thus verifying the Dirac duality relation $e(p)\tilde{e}(p) = 2\pi$ between the effective charges.

At one-loop order, we have $\Pi(p) = Ne_0^2/8p$, implying that at large distances the potential becomes

$$V(R) = -\frac{4}{\pi NR} + \mathcal{O}\left(\frac{1}{R^2}\right), \qquad (24)$$

instead of having the logarithmic behavior expected classically. Thus, quantum fluctuations lead in two spatial dimensions to a large distance behavior of the potential similar to the one of electrodynamics in three spatial dimensions. Interestingly, the above power is $1/R$ for all dimensions $d \in (2,4)$ and the coefficient in front of the factor $1/R$ is universal,[31] a situation similar to the one obtained from string models for the interquark potential.[46]

Since the energy of the instanton gas is changed by the presence of matter, the sine-Gordon theory [Eq. (7)] must be changed correspondingly.[23,27,28] It is easy to see that the field theory of the instanton gas is now given by

$$\mathcal{L} = \frac{e_0^2}{8\pi^2}(\partial_\mu\varphi)Z_A(\sqrt{-\partial^2})(\partial_\mu\varphi) - 2z_0\cos\varphi, \qquad (25)$$

where the symbol $Z_A(\sqrt{-\partial^2})$ has an operator meaning such that in momentum space, the gradient energy reads simply $(e_0^2/8\pi^2)Z_A(p)p^2\varphi(p)\varphi(-p)$. Thus, the Lagrangian in Eq. (25) corresponds to that in Eq. (7) with the bare squared charge e_0^2 replaced by an effective one. Unlike in $\overline{\text{QED}}_{2,1}$ discussed in the previous section, Eq. (25) does not provide an exact dual representation of the theory since its derivation is obtained in a harmonic approximation to the gauge field

fluctuations. However, to the order to which our RG equations were calculated, it provides an accurate dual field theory representation.

After evaluating the momentum integral in Eq. (22) using the short-distance cutoff $a = 1/e_0^2$ and a one-loop approximation to $\Pi(p)$, we obtain the following interaction between two oppositely charged instantons:

$$U_0(r) = -\frac{\pi}{e_0^2}\left[\frac{1}{r} - \frac{1}{a} - \frac{Ne_0^2}{4\pi}\ln\left(\frac{r}{a}\right)\right] = -\frac{\pi}{e_0^2 r} + \frac{N}{4}\ln(e_0^2 r) + \pi,$$
(26)

where $r \equiv |x|$. Let us compute the pair susceptibility at large distances using the above potential. This is obtained from the $r \to \infty$ limit of Eq. (A7) of the Appendix for $d = 3$, and inserting the Boltzmann factor $n(r) \approx z_0^2 e^{-U_0(r)}$. The pair susceptibility obtained in this way gives a measure of $\langle r^2 \rangle$ and is given by

$$\chi_0^* = \lim_{r \to \infty} \chi_0(r) = \frac{16\pi^3 z_0^2}{3e_0^2}\int_a^\infty d\rho \rho^4 e^{-U_0(\rho)},$$
(27)

where the subscript in χ indicates that the calculation is done with the potential $U_0(r)$. By introducing the dimensionless variable $u = e_0^2\rho/\pi$, we obtain

$$\chi_0^* = \frac{16\pi^{8-N/4}e^{-\pi}z_0^2}{3(e_0^2)^6}\int_{1/\pi}^\infty du\, u^{4-N/4} e^{1/u}.$$
(28)

The integral converges only for $N > 20$. This leads to $\varepsilon_0^* = 1 + 4\pi\chi_0^* > 1$, signaling a "dielectric" phase for the instanton gas. Duality implies that the "dielectric" instanton phase corresponds to a "metallic" phase for the spinons, i.e., the spinons are deconfined and we have a spin liquid. On the other hand, for $N \leq 20$, the instantons are in the metallic phase, which corresponds by duality to confined spinons. As we shall see, it is not quite accurate to say that spinons deconfine for $N > 20$. Actually, it will be shown later that in the interval $20 < N \leq 36$, both confinement and deconfinement phases may occur. A stable deconfined phase occurs only for $N > 36$.

The arguments of the preceding paragraph are better understood by a RG analysis. We shall derive the RG equations using the method of the Appendix, with the $U_0(r)$ given in Eq. (26) replacing the one in Eq. (A1). The dielectric constant $\varepsilon(r)$ of the instanton gas as a function of the length scale r has the same form as in the Appendix, except that the renormalized interaction between the instantons is in this case given by

$$U(r) = U(a) + \frac{\pi}{e_0^2}\int_a^r \frac{ds}{\varepsilon(s)s^2}\left(1 + \frac{Ne_0^2}{4\pi}s\right),$$
(29)

which implies a renormalized dimensionless gauge coupling,

$$f(l) = \frac{e^l\varepsilon(ae^l)}{1 + \frac{N}{4\pi}e^l}.$$
(30)

The first RG equation follows immediately from Eq. (30),

$$\frac{df}{dl} = f + y^2 - \frac{N}{4\pi}\frac{f^2}{\varepsilon},$$
(31)

where we have defined

$$y^2(l) = \frac{64\pi^4}{3}\frac{a^6 z_0^2 e^{6l-U(ae^l)}}{1 + \frac{N}{4\pi}e^l}.$$
(32)

The third term in Eq. (31) represents the correction to Eq. (18) due to the presence of fermionic matter. For $y = 0$, the instantons are suppressed and $\varepsilon = 1$, such that Eq. (31) reduces to the β function of noncompact $\text{QED}_{2,1}$. In such a situation, we can rewrite the potential [Eq. (24)] in terms of the fixed point $f* = 4\pi/N$ of the noncompact theory,

$$V(R) = -\frac{f*}{\pi^2 R} + \mathcal{O}\left(\frac{1}{R^2}\right).$$
(33)

By differentiating Eq. (32) with respect to l, we obtain

$$\frac{dy}{dl} = y\left(3 - \frac{\pi}{2f} - \frac{N}{8\pi}\frac{f}{\varepsilon}\right),$$
(34)

such that the correction to Eq. (19) due to fermionic matter is also obtained. The flow of the screening constant of the instanton gas is given by

$$\frac{d\varepsilon}{dl} = \frac{\varepsilon y^2}{f}.$$
(35)

This follows from

$$\frac{d\varepsilon}{dr} = \frac{16\pi^3}{3}\frac{z_0^2 a^4}{e_0^2}e^{4l-U(ae^l)},$$
(36)

which is easily derived using Eqs. (A6) and (A7) of the Appendix.

The last term in Eq. (34) is crucial for understanding QSD from a RG point of view. This term was overlooked before[22–24,28,29] and led to an incomplete physical picture of the phase structure of QSD. In order to emphasize the importance of this term, let us neglect it for a moment. Considering $f* = 4\pi/N$, we find a line of fixed points for zero fugacity as N is varied.[23,24] From this, it was concluded[23,24] that QSD undergoes a KT-like deconfinement transition in three space-time dimensions. In this deconfinement scenario, the critical value of N above which the spinons deconfine is $N_c = 24$. The reason why this term was overlooked before is that it was generally assumed that the logarithm term in Eq. (26) dominates over the three-dimensional (3D) Coulomb interaction at large distances. In other words, the large distance limit was being taken at a too early stage. We may argue that if one has a logarithmic gas of instantons in 3D, it should not be particularly surprising that a 3D KT-like transition emerges. However, to really prove such a statement is far from being trivial and has led to recent controversies.[28–30] Monte Carlo simulations[30] seem to support the scenario of a KT-like transition for the 3D logarithmic gas. However, it is now known that such a transition *does not* describe correctly the deconfinement transition in QSD.[22,31] Here, we shall give

a more complete explanation of the results obtained in Refs. 22 and 31.

In Ref. 22, it was shown that the spinons deconfine for a large but finite N, but the critical value of N above which it happens was not calculated. This value was estimated in our paper[31] to be $N_c = 36/\pi^3$. From the more thorough analysis performed here, we shall see that this estimate lies too low. One trivial reason for this is an overcounting of π factors in the derivation of the RG equations, as was mentioned after Eqs. (18) and (19). However, there are other aspects of the RG flow of QSD which were not considered in Refs. 22 and 31 and will now be included.

From Eqs. (31), (34), and (35), we see that there is apparently for all $N \geq 20$ a *fixed line* at zero fugacity and $f* = 4\pi\varepsilon*/N$, $1 \leq \varepsilon* \leq N/20$. Note that $\varepsilon \geq 1$, since $\chi \geq 0$. However, it is clear that for vanishing fugacity, we must have $\varepsilon* = 1$, so that $N = 20$ and the fixed line actually collapses to a fixed point with $f* = \pi/5$. It is reasonable to assume that the RG functions are given by expansions in the fugacity. For $y = 0$, we have $\varepsilon(l) = 1$ *for all* l, such that Eq. (31) reduces to the RG β function of noncompact QED$_3$. Up to terms of order y^2, we can legitimately approximate $f^2/\varepsilon \approx f^2$ and $yf/\varepsilon \approx yf$ in Eqs. (31) and (34), respectively. In this way, Eqs. (31) and (34) decouple from Eq. (35) and have the following fixed points at nonzero fugacity:

$$f_\pm = \frac{2\pi}{N}(6 \pm \sqrt{36 - N}), \qquad (37)$$

$$y_\pm^2 = \frac{10\pi}{N}\left(6 - \frac{N}{10} \pm \sqrt{36 - N}\right). \qquad (38)$$

We confirm once more the fixed point with zero fugacity at $N = 20$, $y_- = 0$, and $f_- = \pi/5$. The above fixed points do not exist if $N > 36$, in which case only the fixed point of the noncompact theory is available, and the spinons are deconfined. This result confirms the analysis of Ref. 22, where it was argued that spinons deconfine above a large but finite value of N. Thus, for $N > 36$, the theory is well controlled by an expansion in $1/N$. For $N \in (20, 36]$, on the other hand, it seems to correspond to an expansion in the fugacity. In order to investigate this claim, let us consider Eq. (30) in the limit $l \to \infty$, in which case $f \to f_\pm = 4\pi\varepsilon_\pm/N$, where $\varepsilon_\pm = \lim_{l \to \infty} \varepsilon(ae^l)$. From Eqs. (37) and (38), we obtain for $20 < N \leq 36$,

$$\varepsilon_\pm = \frac{N}{20}\left(1 + \frac{1}{\pi}y_\pm^2\right), \qquad (39)$$

and

$$f_\pm = \frac{\pi}{5}\left(1 + \frac{1}{\pi}y_\pm^2\right). \qquad (40)$$

The above equations clearly have the structure of an expansion in y. This provides a further argument for setting $\varepsilon = 1$ in Eqs. (31) and (34). However, it should be noted that within the present approximation, $y_\pm^2$ is not small in the entire interval $20 < N \leq 36$, and perturbation theory may eventually breaks down for some values of N. Most critical is the situ-
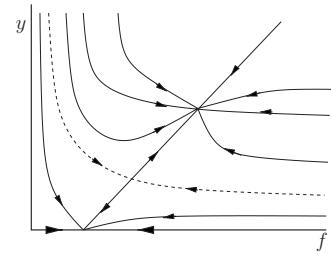

FIG. 1. Schematic flow diagram for $20 < N < 36$ featuring the fixed points at nonzero fugacity given by Eqs. (37) and (38).

ation at the fixed point (f_+, y_+) since y_+ gets small only for N close to 36. Note that also f_+ can be large in this interval. Already at $N = 20$, we have $f_+ = \pi$, indicating that the behavior near the fixed point (f_+, y_+) should be considered with great care. The fixed point (f_-, y_-), on the other hand, has much better asymptotics. Indeed, y_- is small for all values of N in the interval $20 < N < 36$ up to $N = 32$, becoming larger than unity only above this value.

For all $20 < N \leq 36$, the fixed points $(f_\pm, y_\pm)$ govern a confined regime such that the interspinon potential at these fixed points reads

$$V_\pm(R) = \sigma_\pm R - \frac{f_\pm}{\pi^2 R} + V_L(R) + \mathcal{O}(1/R^2), \qquad (41)$$

where $\sigma_\pm$ is the string tension in the presence of fermionic matter. In the present approximation, the string tension has precisely the same form as in the $\overline{\text{QED}}_{2,1}$,[36] except that the bare parameters are replaced by the renormalized ones. Thus, $\sigma = 2e^2M/\pi^2 = 4e\sqrt{2}z/\pi$, or in terms of dimensionless quantities, $\sigma/e_0^4 = 4\sqrt{2}fy/\pi$. The term $V_L(R)$ is assumed to be given by a string model for the electric flux tube due to Lüscher,[46] which in d dimensions has the form

$$V_L(R) = -\frac{(d-2)\pi}{24R}. \qquad (42)$$

Such a term has to be included in order to account for the fluctuations of the electric flux tube occurring even in the absence of matter. The factor $d - 2$ represents the number of transverse modes of the string worldsheet. The coefficients of the $1/R$ terms in Eq. (41) are universal and the string tensions $\sigma_\pm$ reduce to the one obtained by Polyakov[18] in the limit case where no matter fields are present.

The schematic flow diagram for $20 < N < 36$ is shown in Fig. 1. In this case, the system can be either in the confinement or deconfinement phase, since the fixed points on each side of the dashed line in Fig. 1 are infrared stable. The fixed point (f_-, y_-) is infrared stable only along the dashed line in Fig. 1. Precisely, for this reason, it plays an important role in our discussion. The point is that y_- vanishes for $N = 20$, which is the value of N leading to $\varepsilon* = 1$. This result gives a string tension σ_- that vanishes for all $N \leq 20$, thus leading to a collapse of the fixed point (f_-, y_-) into the fixed point of the noncompact theory. Since the fixed point (f_-, y_-) is unstable in the direction leading to the fixed point (f_+, y_+) and the one of the noncompact theory, the latter will no longer be stable

TABLE I. Summary of the phase structure of QSD. The third column gives the universal coefficient of $1/R$.

N	Phase	$-1/R$ coefficient
$N \leqslant 20$	Confined	$\pi/24 + f_+/\pi^2$
$20 < N \leqslant 36$	Confined/deconfined	Like $N \leqslant 20$ or $N > 36$
$N > 36$	Deconfined	$4/\pi N$

and only the confinement phase governed by the infrared stable fixed point (f_+, y_+) will exist. Note that while the string tension σ_- vanishes continuously as N approaches 20 from above, this is not the case as N approaches 36 from below. In fact, $\sigma_\pm / e_0^4 = 4\sqrt{2} f_\pm y_\pm / \pi$ has a finite value for $N = 36$. Since σ vanishes for $N > 36$, it follows that there is a universal jump in the string tension for $N = 36$.

It should be emphasized here that the physical interspinon potential corresponds to the one associated with the infrared stable fixed point, i.e., $V_+(R)$. Since the fixed point (f_-, y_-) is unstable, the dashed line represents a critical line separating confined and deconfined regimes. This role of a separatrix played by the dashed line of Fig. 1 is most clearly seen as N cross the value $N = 20$, in which case the line collapses along the axes f and y, with the fixed point (f_-, y_-) becoming identical to the one of the noncompact theory.

The phase of the system when $20 < N < 36$ is determined by the size of the Debye-Hückel parameter, $\kappa_D = n \lambda_D^3 = \sqrt{2} e^2 / (8\pi^2 M)$, where n is the instanton density and λ_D the Debye length. Since for $N > 36$ only the fixed point at zero fugacity exists, we have that $\sigma_+ = 0$, i.e., deconfinement occurs for $N > 36$ and the large-N result is essentially correct.[22]

In Table I, we summarize the phase structure of QSD.

IV. DISCUSSION

We have presented in this paper a description of the quantum spinon dynamics in a Heisenberg antiferromagnet using a field theory of N identical replica of fermions with a constraint. Only the limit $N \to \infty$ can be treated exactly, the lower N results are obtained from different approximations. Since $N = 2$ corresponds to the physically interesting case, one may wonder to which extent our results are physically relevant. This question is hard to answer. Originally, the study of spin liquids in 2+1 dimensions was motivated by the properties of high-T_c cuprate superconductors.[10] It was hoped that doping a Mott insulator provides a mechanism for superconductivity in the cuprates. At zero doping, the cuprates are antiferromagnetic insulators, i.e., the global SU(2) spin symmetry of the model is broken in the ground state. Spin liquids, on the other hand, have no broken symmetries. Nevertheless, by doping a spin liquid, we expect to obtain a superconducting state.[14,47] The idea was that doping may frustrate the magnetic state of the system and produce a spin liquid. With further doping the system would eventually become superconducting through a kind of hole condensation. Hence, doping plays a role similar to frustration due to the triangular lattice in the organic compound κ-$(ET)_2Cu(CN)_3$, where a spin liquid phase was recently reported.[48]

Our analysis considers neither doping nor triangular lattices. Instead, the spin liquid emerges from the large-N limit, which provides the needed frustration to produce a spin liquid. Our treatment at lower values of N was based on an expansion in the instanton fugacity. This treatment is essentially nonperturbative, and as such difficult to control. Nevertheless, the stability of the spin liquid for large N seems to be established beyond doubt.

Note, however, that the large-N analysis of the lattice model is made in a particular representation of the SU(N) group by writing the spin operators in terms of fermions.[5,19] If the spin operators are written in terms of bosons—which is an equally valid description of the problem—other representations are obtained. In such a case the Berry phase plays an essential role in determining the phases of the system.[49] There N is kept fixed and large, while a coupling constant g related to the exchange constant is varied. In the lower phase, i.e., for $g < g_c$, where g_c is the critical coupling, the SU(N) symmetry is broken and we have a Néel-like state. For $g > g_c$, on the other hand, the paramagnetic phase *is not* a spin liquid. Instead, the so-called valence-bond solid (VBS) state emerges.[49] For $N = 2$, the VBS state can only occur if frustrating interactions respecting the SU(2) symmetry are included in the Hamiltonian. In such a scenario, where also an effective compact U(1) gauge theory is featured, the spinons are confined in both phases, but conjectured to be deconfined precisely at $g = g_c$. This is the so-called *deconfined quantum criticality scenario* and provides a new paradigm for quantum phase transitions.[50] One of the main characteristics of the this new type of quantum criticality is a large value for the η exponent. In which systems, this new paradigm actually holds is still under discussion. There is a recent numerical evidence[51] supporting the scenario proposed in Ref. 50. On the other hand, it was recently demonstrated using large scale Monte Carlo simulations[52] that quantum antiferromagnets with easy-plane anisotropy do not exhibit deconfined quantum criticality, contrary claims in Ref. 50.

In QSD, our analysis certainly breaks down for $N = 2$. The description in terms of the generalized sine-Gordon theory [Eq. (25)] no longer holds. The reason for this is that the derivation of the Lagrangian [Eq. (25)] assumes that the fermions are massless,[23,27,28] and it is known that in noncompact $QED_{2,1}$, the fermion mass is dynamically generated through chiral symmetry breaking (CSB) for small N.[3,4] While the precise value of N below which CSB occurs is still a matter of debate, it seems to be more or less consensual that the chiral symmetry is broken for $N = 2$.[53] Massive fermions modify the vacuum polarization in such a way that makes the derivation of an effective instanton Lagrangian difficult. Furthermore, independent of the CSB occurring in noncompact $QED_{2,1}$, confinement is likely to induce CSB in QSD already for N below and near 20, in a regime where the effective Lagrangian [Eq. (25)] can still be considered to be valid. In any case, CSB corresponds to spin density waves and this is precisely the state one would expect for undoped cuprates. The conclusion seems to be that no spin liquid state is possible for $N = 2$ in the undoped system, unless frustrating interactions are added. In this case, it may well be possible that a VBS emerges instead a spin liquid. Moreover, the resulting paramagnetic state may depend on the nature of the

frustration. Here, it is worth mentioning that a local SU(2) gauge theory of (fermionic) spinons for a frustrated Heisenberg antiferromagnet[16] exhibits a stable spin liquid phase.

When doping is included the RG analysis becomes considerably more difficult due to the coupling of the gauge field with a nonrelativistic complex scalar field.[14] In this case, there are additional topological defects. These are vortex excitations coupled to the instantons. We are currently investigating this situation.[54]

APPENDIX: RENORMALIZATION GROUP EQUATIONS FOR THE *d*-DIMENSIONAL COULOMB GAS

In order to make the paper self-contained, we consider here the general *d*-dimensional Coulomb gas, whose RG equations were set up by Kosterlitz.[45] They were originally derived using the poor-man scaling approach. Here, we employ the method due to Young,[44] which is physically appealing, since it amounts to applying a scale-dependent Debye-Hückel argument which leads to the same results. Although Young applied the method to derive the RG equations associated to the KT phase transition, it can easily be generalized to the *d*-dimensional case. We have done this previously[27] to derive the RG equations for anomalous Coulomb gases in *d* dimensions. Here, we concentrate on the ordinary *d*-dimensional Coulomb gas.

The bare Coulomb interaction between two opposite charges of magnitude $\sqrt{K_0}$ is given by

$$U_0(r) = -4\pi^2 K_0 V(r), \tag{A1}$$

where

$$V(r) = \frac{a^{2-d}}{4\pi^{d/2}} \Gamma\left(\frac{d}{2}-1\right) \left[\left(\frac{r}{a}\right)^{2-d} - 1\right]. \tag{A2}$$

In the above equation, a is a short-distance cutoff, which for $d=3$ will be set to $a=1/e_0^2$. From Eq. (A1), we obtain the bare electric field,

$$E_0(r) = -4\pi^2 c(d)\frac{K_0}{r^{d-1}}, \tag{A3}$$

where

$$c(d) = \frac{d-2}{4\pi^{d/2}} \Gamma\left(\frac{d}{2}-1\right). \tag{A4}$$

Next, in the spirit of the Debye-Hückel theory, we introduce an effective medium via a scale-dependent dielectric constant $\varepsilon(r)$. This gives the renormalized electric field,

$$E(r) = -4\pi^2 c(d)\frac{K_0}{\varepsilon(r)r^{d-1}}. \tag{A5}$$

The dielectric constant $\varepsilon(r)$ is expressed in terms of the susceptibility $\chi(r)$ as

$$\varepsilon(r) = 1 + S_d \chi(r), \tag{A6}$$

where

$$\chi(r) = S_d \int_a^r ds\, s^{d-1}\alpha(s)n(s), \tag{A7}$$

with $S_d = 2\pi^{d/2}/\Gamma(d/2)$ being the surface of the unit sphere in d dimensions, and $\alpha(r)$ is the polarizability. For small separation of a dipole pair, it is given approximately by

$$\alpha(r) \approx 4\pi^2 K_0 r^2/d. \tag{A8}$$

The average number of dipole pairs is

$$n(r) \approx z_0^2 e^{-U(r)}, \tag{A9}$$

where z_0 is the bare fugacity and $U(r)$ the renormalized potential obtained by integrating the renormalized electric field [Eq. (A5)],

$$U(r) = U(a) + 4\pi^2 c(d)K_0 \int_a^r \frac{ds}{s^{d-1}\varepsilon(s)}. \tag{A10}$$

The renormalized version of K_0 is

$$\frac{1}{K(l)} = \frac{\varepsilon(ae^l)}{K_0} e^{(d-2)l}, \tag{A11}$$

where $l = \ln(r/a)$. Differentiating Eq. (A10) with respect to l, and using Eq. (A11), yields

$$\frac{dU}{dl} = \frac{4\pi^2 c(d)}{a^{d-2}} K(l), \tag{A12}$$

Next, we differentiate Eq. (A11) with respect to l to obtain

$$\frac{dK^{-1}}{dl} = \frac{8\pi^2 S_d^2 z_0^2 a^{d+2}}{d} e^{2dl-U(ae^l)} + (d-2)K^{-1}. \tag{A13}$$

Here, we define

$$z^2(l) = \frac{8\pi^2 S_d^2 z_0^2}{d} e^{2dl-U(ae^l)}, \tag{A14}$$

such that Eq. (A13) becomes

$$\frac{dK^{-1}}{dl} = a^{d+2}z^2 + (d-2)K^{-1}. \tag{A15}$$

From Eq. (A14), we derive the RG equation for the effective fugacity,

$$\frac{dz}{dl} = \left[d - \frac{2\pi^2 c(d)K}{a^{d-2}}\right]z. \tag{A16}$$

It is convenient to introduce the dimensionless quantities $\kappa \equiv a^{2-d}K$ and $y \equiv a^d z$ to rewrite Eqs. (A15) and (A16) as

$$\frac{d\kappa^{-1}}{dl} = y^2 + (d-2)\kappa^{-1}, \tag{A17}$$

$$\frac{dy}{dl} = [d - 2\pi^2 c(d)\kappa]y. \tag{A18}$$

For $d=2$, the above RG equations govern the scaling behavior of the KT transition, while for $d>2$, there is no fixed point, implying that the d-dimensional Coulomb gas is always in the metallic phase.[45]

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